International Journal of Theoretical Physics, Vol. 29, No. 10, 1990

Virasoro Algebras in the Theory of Bosonic *p*-Branes

R. P. Zaikov[1]

Received April 24, 1990

It is shown that in the case of closed bosonic p-branes there exist $p+1$ mutually commuting Virasoro algebras which are a direct generalization of the string case. The existence of these algebras allows us to conclude that a non-Abelian string spectrum and quantum anomalies are admissible. The generalization of these results for the supersymmetric case is also discussed.

Following the pioneering work of Dirac (1962), the membrane theory was considered also in connection with the bag theory (Collins and Tucker, 1976). In recent years the interest in membranes and in general in p-branes (p-dimensional extended objects) has been renewed due to a possible generalization of the string theory (Kikkawa and Yamasaki, 1986; Kubo, 1987; Bergshoeff *et al.*, 1987; Duff *et al.*, 1987). It is important that the Lagrangian of the p-branes (when $p>1$) contains an essential nonlinearity, i.e., there exist self-interactions as in the Yang-Mills theory. Consequently, it is not surprising that in the membrane ($p=2$) and p-brane ($p>2$) theories, new difficulties arise which are absent in string theories. One such difficulty is that the corresponding constraint algebra does not form a closed Lie algebra when $p>1$. The latter was the reason for some people to claim that the membrane theory was free from anomalies. Recently some restricted reparametrization transformations were considered by Floratos and Iliopoulos (1988), Antoniadis *et al.* (1988), and Bars *et al.* (1988), the so-called area-preserving transformations, which form a closed Lie algebra. The latter diffeomorphisms are the residual symmetry of the p-brane action in the light-cone gauge. By some manipulations from the latter only one copy of the Virasoro algebra was found. However, such area-preserving transformations do not exist in the string case and, moreover, the string theories contain two copies of the Virasoro algebra.

[1]Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria.

0020-7748/90/1000-1059$06.00/0

In Zaikov (1988) some extension of the area-preserving transformations was considered which gives the Virasoro algebras in the string case as well as for the 3-branes. In the present paper we consider another subset of diffeomorphisms which are a residual symmetry of the p-brane action in the orthogonal gauge. In the string case we have two mutually commuting Virasoro algebras. This result is generalized for arbitrary p-branes for which $p+1$ copies of mutually commuting Virasoro algebras with the corresponding central terms are derived. An additional $SO(p, 1)$ symmetry is discussed which makes clear the non-Abelian character of the p-brane ($p>$) theories. Let us point out that our considerations are also directly applicable for the supermembrane theory (Lindstrom and Rocek, 1988).

Let us consider the Nambu-Goto action for the closed classical bosonic p-branes:

$$S = T\int d\tau\, d^p\sigma\, L(X) = T\int d\tau\, d^p\sigma \sqrt{g} \tag{1}$$

where T is a parameter with dimension of $[\text{mass}]^{p+1}$, $g=\det(g_{ab})$, $g_{ab}=X^{\mu}_{,a}X_{\mu,b}$ ($\mu=0,1,\ldots,D-1$; $a,b=1,\ldots,p+1$) is the induced metric, $X^{\mu}_{,a}=\partial X^{\mu}/\partial\xi^{a}$, $\xi=(\tau,\sigma_1,\ldots,\sigma_p)$, and the functions $X^{\mu}(\tau,\sigma)$ are the embedding of the p-brane Σ_p in the D-dimensional Minkowski space-time. For the closed p-brane we have the ordinary periodicity condition

$$X^{\mu}(\tau,\sigma+2\pi)=X^{\mu}(\tau,\sigma) \tag{2}$$

The action integral (1) is invariant with respect to general reparametrization transformations

$$\xi_a \Rightarrow \eta_a(\xi) \tag{3}$$

As a consequence of this invariance, we have the following constraints:

$$\begin{aligned} &\Phi_j = X^{\mu}_{,j}\mathcal{P}_{\mu}=0 \\ &\Phi_{p+1}=(\mathcal{P}_{\mu})^2-\det(g_{jk})=0 \qquad (j,k=1,\ldots,p) \end{aligned} \tag{4}$$

where

$$\mathcal{P}_{\mu}=\partial L/\partial X^{\mu}_{,\tau} \tag{5}$$

is the canonical conjugate momentum. It is easy to verify that the constraints (4) with respect to the Poisson brackets do not form a Lie algebra if $p>1$. Recall that when $p=1$ the corresponding algebra is just the Virasoro algebra. Let us recall also that the constraints (4) are generators of the infinitesimal reparametrization transformations (3).

Now, following the strategy of Floratos and Iliopoulos (1988), Antonaidis *et al.* (1988), Bars *et al.* (1988), and Zaikov (1988), we consider some restricted class of infinitesimal diffeomorphisms (3)

$$\xi_a \Rightarrow \xi_a + \delta\omega_a h_a(\xi) \qquad (a=0,1,\ldots,p) \tag{6}$$

where there is no summing among the repeated indexes a, $\delta\omega_a$ are constant infinitesimal parameters, and h_a are arbitrary functions satisfying the boundary conditions (2). In the papers of Floratos and Iliopoulos (1988), Antoniadis *et al.* (1988), Bars *et al.* (1988), and Zaikov (1988) these functions were determined from the equation

$$\partial^a h_a(\xi) = 0 \tag{7}$$

which coincides with the so-called area-preserving condition if

$$\delta\omega_0 = \delta\omega_1 = \cdots = \delta\omega_p = \delta\omega \tag{8}$$

More exactly, in Floratos and Iliopoulos (1988) and Antoniadis *et al.* (1988) the membrane case in the light-cone gauge and in Bars *et al.* (1988) the p-branes in the light-cone gauge were considered in which $h_j = h_j(\sigma_1, \ldots, \sigma_p)$. Then it is evident that the generators

$$\mathscr{L} = h_j \partial_j \tag{9}$$

in the string case do not form an infinite-dimensional Lie algebra, because $h_1 = \text{const}$. To make this possible (Zaikov, 1988), the diffeomorphisms (6) depending also on the evolution parameter τ were considered and the condition (8) relaxed.

Now, let us consider the following generators:

$$\mathscr{L}_a = h_a(\xi)\partial_a \qquad (a = 0, 1, \ldots, p) \tag{10}$$

where there is no summing among the repeated indices a, and h_a are general diffeomorphisms (7). The generators (10) satisfy the following Lie algebra:

$$[\mathscr{V}_a^h, \mathscr{V}_b^g] = \mathscr{V}_b^{\{h,g\}} - \mathscr{V}_a^{\{g,h\}} + f(h, g) \tag{11}$$

Here $\{h, g\} = h_a \partial_a g_b$, $\mathscr{V}$ are the quantum generators corresponding to (10) and

$$f(h, g) = \int d^{p+1}\xi \, \Omega(\xi)(h\partial_a g - g\partial_b h) \tag{12}$$

where $\Omega(\xi)$ is an arbitrary function. The central charge term f is determined from the Jacobi identity and indicates that there are possible anomalies in the quantum case (Bars *et al.*, 1988).

To specify the algebra (11), let us consider first the string case. The problem is to find such diffeomorphisms (6) for which the algebra (11) reduces to the Virasoro algebra. In Zaikov (1988) this was achieved using the solutions of equation (7). However, in the p-brane case such diffeomorphisms are very restrictive. For this reason, let us consider the string Lagrangian in orthogonal gauge,

$$L = (X_{,0}^2 X_{,1}^2)^{1/2} \tag{13}$$

It is evident that the action corresponding to (13) is invariant with respect to the following restricted reparametrizations:

$$\tau \Rightarrow \tau + \delta\omega_0 u(\tau), \qquad \sigma \Rightarrow \sigma + \delta\omega_1 v(\sigma) \tag{14}$$

Then inserting in (10) the generating functions (6) corresponding to (14), we have

$$\mathscr{L}_0^u = u(\tau)\partial_\tau, \qquad \mathscr{L}_\sigma^v = v(\sigma)\partial_\sigma \tag{15}$$

for which we obtain the Virasoro algebras

$$\begin{aligned} [\mathscr{V}_0^{u_1}, \mathscr{V}_0^{u_2}] &= \mathscr{V}_0^{u_{12}} + f_0(u_1, u_2) \\ [\mathscr{V}_1^{v_1}, \mathscr{V}_1^{v_2}] &= \mathscr{V}_1^{v_{12}} + f_1(v_1, v_2) \\ [\mathscr{V}_0^u, \mathscr{V}_1^v] &= 0 \end{aligned} \tag{16}$$

Here $u^{12} = u^1 u^2_{,\tau} - u^2 u^1_{,\tau}$, $v^{12} = v^1 v^2_{,\sigma} - v^2 v^1_{,\sigma}$, and the central terms appearing in the quantum case are determined from the Jacobi identity, which reads:

$$f(u^{12}, u^3) + f(u^{31}, u^2) + f(u^{23}, u^1) = 0 \tag{17}$$

The general solution of (17) is given by (Bars *et al.*, 1988)

$$f_a(u, v) = \int d\xi_a \, \{\mathscr{A}(\xi_a)(uv_{,a} - vu_{,a}) + \mathscr{B}(\xi_a)(uv_{,a} - vu_{,a})_{,aa}\} \tag{18}$$

where $\mathscr{A}$ and $\mathscr{B}$ are arbitrary functions of the single variable ξ_a and $u_{,a} = du/d\xi_a$.

In terms of plane waves, the Virasoro generators (14) are written as

$$\mathscr{L}_0^m = -i\, e^{im\tau}\partial_\tau, \qquad \mathscr{L}_1^m = -i\, e^{im\sigma}\partial_\sigma \tag{19}$$

It is evident that $\mathscr{L}_{0,1}$ form two mutually commuting Virasoro algebras

$$[V^m, V^n] = (m-n)V^{m+n} + cm(m^2-1)\delta_{m+n} \tag{20}$$

Now, let us introduce a (pseudo)orthogonal matrix U ($UU^T = U^T U = I$). Then consider the transformed generators (19):

$$\mathscr{L}_a^m = -i\, e^{im(U\xi)_a}\partial_{(U\xi)_a} \qquad (a = 0, 1) \tag{21}$$

which satisfy the same algebra (20). Consequently, we have additional Abelian $SO(1, 1)$ [$SO(2)$ in the Euclidean case] symmetry. In the special case

$$U = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we find from (21) the well-known Virasoro generators in terms of light-cone variables.

Now, let us consider the general case of p-branes ($p > 1$). The one-variable reparametrizations corresponding to (14) are given by

$$\delta\xi_0 = \delta\omega_0 u_0(\xi_0), \qquad \delta\xi_1 = \delta\omega_1 u_1(\xi_1), \ldots, \qquad \delta\xi_p = \delta\omega_p u_p(\xi_p) \tag{22}$$

Then, inserting (22) in (10), we find the generators of $p+1$ copies of mutually commuting Virasoro algebras:

$$[\mathcal{V}_a^{u_a}, \mathcal{V}_b^{v_b}] = \delta_b^a(\mathcal{V}_a^{\{u,v\}} + f_a(u, v)) \tag{23}$$

where the central charge term f has the same form as (18). Inserting in (23) the plane wave expansion in the case when $\Sigma = T_p$, where T_p is a p-dimensional torus, we have $p+1$ copies of the Virasoro algebra (20). If $U \in SO(p, 1)$, then in the same way as in the string case [see equation (21)], from (23) we find another set of Virasoro generators. The last result allows us to conclude that there exists an additional $SO(p, 1)$ symmetry. Consequently, in the membrane case as well as in the general b-brane case this symmetry is non-Abelian. The latter is not surprising, because the membrane (p-brane) Lagrangian contains essential nonlinearities as in the Yang–Mills theory. This non-Abelian Virasoro symmetry confirms the existence of a non-Abelian string spectrum in the p-brane theory (Kubo, 1987), and it points out the possibility for the arising of quantum anomalies.

We remark that there is a more general type of Lie algebra (11); however, according to (12), they admit only linear central charge (Floratos and Iliopoulos, 1988; Antoniadis *et al.*, 1988; Bars *et al.*, 1988).

Finally, let us point out that our considerations can also be generalized to the supersymmetric case. This can be seen from (23) because any copy of the Virasoro algebra can be extended to the corresponding super-Virasoro algebra with the ordinary central charge. The latter contradicts the results of Bars *et al.* (1988), where from the absence of a central term in the area-preserving super Lie algebra it was concluded that the quantum anomalies did not exist in the supersymmetric p-brane theory with world-volume supersymmetry considered from Lindstrom and Rocek (1988).

ACKNOWLEDGMENT

It is a pleasure to thank Dr. S. Pacheva for a critical reading of the manuscript.

REFERENCES

Antoniadis, I., Ditsas, P., Floratos, E., and Iliopoulos, J. (1988). CERN preprint TH 4970/88, Geneva.

Bars, I., Pope, C. N., and Sezgin, E. (1988). *Physics Letters B*, **210**, 85.

Bergshoeff, E., Sezgin, E., and Townsend, P. K. (1987). *Physics Letters B*, **189**, 75.
Collins, P. A., and Tucker, R. W. (1976). *Nuclear Physics B*, **112**, 150.
Dirac, P. A. M. (1962). *Proceedings of the Royal Society A*, **268**, 57.
Duff, M. D., Hove, P. S., Imami, T., and Stelle, K. S. (1987). *Physics Letters B*, **191**, 70.
Floratos, E. G., and Iliopoulos, J. (1988). *Physics Letters B*, **201**, 237.
Kikkawa, K., and Yamasaki, M. (1986). *Progress of Theoretical Physics*, **76**, 1379.
Kubo, J. (1987). Max Planck Institute preprint, MPI-PAE/83/87, Munich.
Lindstrom, U., and Rocek, M. (1988). SUNY preprint ITP-SB-88-61, Stony Brook, New York.
Zaikov, R. P. (1988). *Physics Letters B*, **213**, 42*y*.